

# Normalizability analysis of the generalized quantum electrodynamics from the causal point of view

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## Abstract

The causal perturbation theory is an axiomatic perturbative theory of the S-matrix. This formalism has as its essence the following axioms: causality, Lorentz invariance and asymptotic conditions. Any other property must be showed via the inductive method order-by-order and, of course, it depends on the particular physical model. In this work we shall study the normalizability of the generalized quantum electrodynamics in the framework of the causal approach. Furthermore, we analyse the implication of the gauge invariance onto the model and obtain the respective Ward-Takahashi-Fradkin identities.

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## 1 Introduction

In the usual approach, the main idea behind renormalization program is that ultraviolet (UV) divergences of a field theory are to be absorbed by an appropriated renormalization of the parameters of the theory [1–3].<sup>1</sup> For perturbative field theories this process is performed order-by-order for each class of graphs. Besides, it is known that the majority of physical field theories, described by first-order Lagrangians, are usually plagued with ultraviolet and infrared divergences. More importantly, it is possible to know whether the renormalization program is applicable to a particular model; for a perturbative theory this can be determined by the method of dimensional analysis and power-counting [4]. Thus, we say that a theory is renormalizable if all the necessary counterterms are found directly from the original Lagrangian.

It is a well known fact that higher-order derivative (HD) theories [5] have, in light of effective field theory [6], better renormalizability properties than the conventional ones. This idea is rather successful in the case of an attempt to quantize gravity, where the (non-renormalizable) Einstein action is supplied by terms containing higher powers of curvature leading to a renormalizable [7] and asymptotically free theory [8]. Also, a new impetus in exploring appealing quantum gravitational theories, such as  $f(R)$ -gravity [9] and Horava-Lifshitz gravity [10]. However, it was soon recognized

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<sup>1</sup>Infrared (IR) divergences may as well appear and these come from the existence of long-range forces in the theory.

that HD theories have a Hamiltonian which is not bounded from below [11] and that the addition of such terms leads to the existence of negative norm states (or ghosts states) – induces an indefinite metric in the space of states – jeopardizing thus the unitarity [12]. Despite the fact that many attempts to overcome these ghost states have been proposed, no one has been able to give a general method to deal with them [13, 14].

One of the most interesting contributions to show the effectiveness of the HD terms in field theory were due to Bopp [15] and Podolsky and Schwed [16], who proposed a generalization of the Maxwell electromagnetic field.<sup>2</sup> The quantum-particle of this field is called Podolsky photon and the interaction of these quanta with electrons is known as generalized quantum electrodynamics (GQED<sub>4</sub>). Moreover, in Ref. [20], it has been shown that the Podolsky Lagrangian is the only linear generalization of Maxwell electrodynamics that preserves invariance under  $U(1)$ . Recently, in Ref. [21] a procedure was suggested for including interactions in free HD systems without breaking their stability. Remarkably, they showed that the dynamics of the GQED<sub>4</sub> is stable at both classical and quantum level.

It is worth to mention that GQED<sub>4</sub> have momentous difference with respect to QED<sub>4</sub>. One of these aspects is the expression of the Born approximation of Bhabha scattering [22], where the Podolsky mass plays a part of a cut-off term for this process. Furthermore, in a previous work [23] the renormalization program was successfully applied on GQED<sub>4</sub>; subsequent quantities were computed at one-loop approximation and showed that the self-energy and vertex are UV finite. Besides, a discussion and evaluation of the Podolsky contribution to the electron anomalous magnetic moment were addressed in detail.

Nonetheless, there exists an alternative and richer approach to analyse the renormalizability property of a physical model. This is the causal perturbation theory, or simply Epstein-Glaser causal theory [24]. In this framework the concept of normalizability is introduced, which is somehow analogous to the usual renormalizability. The main difference between them is that in the causal approach every Green's function is finite order-by-order, thus, the normalization is not a process to subtract divergences, but rather to fix some finite constants.

The causal method is an axiomatic perturbative approach used to systematically compute the elements of the S-matrix [24, 25], by following closely the Heisenberg program [26]. The causal method takes into account only asymptotic free conditions and some few general properties: causality and Lorentz invariance. A remarkable advantage of this approach is that any quantity is defined within the framework of distribution theory; hence, all the product of field operators are well-defined at the origin or, equivalently, they are UV finite [27]. This means that no regularization method is necessary to be introduced [28].

The causal approach has been applied in the analysis to prove the renormalizability of QED<sub>4</sub> [25] and QED<sub>3</sub> [29]. Also in QED<sub>3</sub>, the Epstein-Glaser theory was used to give a clear solution of

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<sup>2</sup>A non-Abelian version of the Bopp-Podolsky electrodynamics was studied and deeply analyzed in [17, 18] as well its interaction with the gravitational field [19].

the Pauli-Villar regularization problem [30]. In a similar way, the causal approach helped to prove inductively the normalizability and gauge invariance of the scalar quantum electrodynamics (SQED<sub>4</sub>) [31]. Furthermore, this framework has also been employed in the study of the gauged Thirring model [32], where the nonrenormalizability of the model and its dynamical mass generation were proved. In all the stated cases the analysis of the divergence nature of higher order graphs has shown to be rather clear and strong than the usual (naive) power counting method.

It should be emphasized, however, that the study of normalizability for these models is possible because the causal approach follows an inductive method to construct any element of the S-matrix series. So, we can analyze the properties of the particular model for each one of these elements, in particular its singularity degree. Subsequently, we determine unambiguously how many constants we need to fix or normalize. In this work we shall use the causal method to study the normalizability of the GQED<sub>4</sub> and the consequence of it onto its gauge invariance. Also, the Ward-Takahashi-Fradkin identities are perturbatively determined.

The work is organized as follows. In Sec. 2, we briefly review the Bopp-Podolsky and Dirac fields, we also give some essentials properties of these fields necessary to develop the inductive method. In Sec. 3, we summarize the main points of the Epstein-Glaser method in order to establish the inductive method. In Sec. 4, we demonstrate the normalizability of the GQED<sub>4</sub> by determining the singular order of each term of the perturbative S-matrix series. In Sec. 5, we use the previous obtained results to show, from the causal point of view, the gauge invariance of the GQED<sub>4</sub> and also that the Ward-Takahashi-Fradkin identities are consistently satisfied order-by-order. Finally, our conclusions and remarks are given in Sec. 6.

## 2 Podolsky's electrodynamics

The lepton-photon interaction can be described by the generalized quantum electrodynamics (GQED<sub>4</sub>), which is endowed with a local  $U(1)$  gauge invariance. Thus, the GQED<sub>4</sub> is described by the Lagrangian

$$\mathcal{L}_{GQED} = \mathcal{L}_D + \mathcal{L}_P + \mathcal{L}_{int}. \quad (2.1)$$

The quantities  $\mathcal{L}_D$ ,  $\mathcal{L}_P$  are, respectively, the free Lagrangians that describe the Dirac and Bopp-Podolsky electromagnetic free fields and define their propagators, at last,  $\mathcal{L}_{int}$  is the interaction part.

Moreover, we have that the dynamics of the free Dirac fields ( $\psi, \bar{\psi} = \psi^\dagger \gamma_0$ ) is governed by  $\mathcal{L}_D = \bar{\psi}(i\gamma \cdot \partial - m)\psi$ . Hence, these fields satisfy the free Dirac equations

$$(i\gamma \cdot \partial - m)\psi = 0, \quad \bar{\psi}(i\gamma \cdot \overleftarrow{\partial} + m) = 0, \quad (2.2)$$

where  $m$  is the physical mass of the fermions. All the parameters are (asymptotically) physical within the causal approach. Using the analytic representation for the propagators [22], we can find the positive (PF) and negative (NF) frequency fermionic propagators

$$\hat{S}^{(\pm)}(p) = (\gamma \cdot p + m) \hat{D}_m^{(\pm)}(p), \quad (2.3)$$

where  $\hat{D}_m^{(\pm)}(p) = \pm \frac{i}{2\pi} \theta(\pm p_0) \delta(p^2 - m^2)$  are the PF and NF massive scalar propagators. These propagators are related to the contraction of the operator fields as follows

$$\overbrace{\psi_e(x) \bar{\psi}_f(y)} = \frac{1}{i} S_{ef}^{(+)}(x-y), \quad \overbrace{\bar{\psi}_e(x) \psi_f(y)} = \frac{1}{i} S_{fe}^{(-)}(y-x). \quad (2.4)$$

On the other hand, for the Bopp-Podolsky electromagnetic field we consider that it is described by the following gauge-fixed Lagrangian [22]

$$\mathcal{L}_P = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial_\mu F^{\mu\sigma} \partial^\nu F_{\nu\sigma} - \frac{1}{2\xi} (\partial \cdot A) (1 + a^2 \square) (\partial \cdot A), \quad (2.5)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the field strength,  $a$  is the Bopp-Podolsky parameter, and  $\xi$  is the constant gauge-fixing parameter. The gauge-fixing term was introduced via the Lagrange multiplier method, where we have chosen to work with the *non-mixing gauge condition* [33]

$$(1 + a^2 \square)^{1/2} \partial^\mu A_\mu = 0, \quad (2.6)$$

which is a pseudodifferential equation [34]. It is important to emphasize that for the non-mixing gauge condition we can obtain, for the choice  $\xi = 1$ , the following field equation:

$$(1 + a^2 \square) \square A_\mu = 0. \quad (2.7)$$

Hence, from this expression, we can affirm that the Bopp-Podolsky field has two non-mixing sectors

$$\square A_\mu^M = 0, \quad (\square + m_a^2) A_\mu^P = 0, \quad (2.8)$$

a massless and massive propagating modes (see (2.9)), i.e. Maxwell and Proca sectors, respectively. The mass of the photon in the Proca sector is given by:  $m_a = a^{-1}$ . Further on, again using the analytic representation [22], we can find the PF and NF electromagnetic propagators

$$\begin{aligned} \hat{D}_{\mu\nu}^{(\pm)}(k) &= g_{\mu\nu} \left( \hat{D}_0^{(\pm)}(k) - \hat{D}_{m_a}^{(\pm)}(k) \right) - (1 - \xi) k_\mu k_\nu \hat{D}_0'^{(\pm)}(k) \\ &+ (1 - \xi) \frac{k_\mu k_\nu}{m_a^2} \left( \hat{D}_{m_a}^{(\pm)}(k) - \hat{D}_0^{(\pm)}(k) \right), \end{aligned} \quad (2.9)$$

where  $\hat{D}_0^{(\pm)}$  and  $\hat{D}_{m_a}^{(\pm)}$  are the PF and NF scalar propagators for the massless and massive modes, respectively; in particular,  $\hat{D}_0'^{(\pm)}(k) = \mp \frac{i}{2\pi} \theta(\pm k_0) \delta(k^2)$  are the PF and NF dipolar massless scalar propagators. The PF electromagnetic propagator is related to the following contraction of the electromagnetic fields

$$\overbrace{A_\mu(x) A_\nu(y)} \equiv \left[ A_\mu^{(-)}(x), A_\nu^{(+)}(y) \right] = i D_{\mu\nu}^{(+)}(x-y). \quad (2.10)$$

Finally, according to the minimal coupling method, we have that the interaction Lagrangian,  $\mathcal{L}_{int}$ , is given by

$$\mathcal{L}_{int} = e : \bar{\psi}(x) \gamma^\mu \psi(x) : A_\mu(x), \quad (2.11)$$

where  $:$  indicates the normal ordering and  $e$  is the constant coupling (normalized, in the causal approach). Also, we can identify  $j^\mu(x) =: \bar{\psi}(x) \gamma^\mu \psi(x) :$  as the electromagnetic current.

### 3 The S-matrix's inductive causal program

In this section, the construction of a perturbative quantum field theory is reviewed from the point of view of the Epstein-Glaser causal theory [24]. In this approach the S-matrix is constructed with no reference to the Hamiltonian formalism, rather it consider an axiomatic formulation. The causal approach postulates the S-matrix in the following formal perturbative series

$$S[g] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 dx_2 \dots dx_n T_n(x_1, x_2, \dots, x_n) g(x_1) g(x_2) \dots g(x_n), \quad (3.1)$$

where we can identify the quantity  $T_n$  as an operator-valued distribution and  $g^{\otimes n}$  its test function. In order to guarantee the existence of (momentum) Fourier transformed expressions it is considered that the test function  $g$  belongs to the Schwartz space  $\mathcal{S}(M^4)$ .<sup>3</sup> In a similar way, we have that the inverse S-Matrix has the form [1]

$$S^{-1}[g] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 dx_2 \dots dx_n \tilde{T}_n(x_1, x_2, \dots, x_n) g(x_1) g(x_2) \dots g(x_n), \quad (3.2)$$

where the distributions  $\tilde{T}_n$  can be obtained by a formal inversion of (3.1). Then, we can find the relation [25]

$$\tilde{T}_n(X_n) = \sum_{r=1}^n (-1)^r \sum_{P_r} [T_{n_1}(X_1) \dots T_{n_r}(X_r)], \quad (3.3)$$

where the sum runs over all partitions  $P_r$  of  $\{x_1, \dots, x_n\}$  into non-empty  $r$  disjoint sets:  $X_n = \bigcup_{j=1}^r X_j$ , with  $X_j \neq \emptyset$  and  $|X_j| = n_j$ .

In this axiomatic approach, the construction of the building blocks  $T_n$  is given via the inductive method. This method is determined when we consider, as postulates, the general physical principles of *causality* [35], *relativistic invariance* [36], and the *asymptotic conditions* in the sense of Heisenberg's program [26]. Since in this approach the S-matrix is a functional of the test function  $g$  [1], these postulates can easily be introduced:

- *Causality*, this principle is understood as the possibility of localizing and ordering events in the space-time. Thus, it can be formulated by the causal ordering relation

$$S[g_1 + g_2] = S[g_2] S[g_1], \quad \text{if } \text{Supp}(g_1) < \text{Supp}(g_2), \quad (3.4)$$

when we substitute this into the perturbative S-matrix series (3.1), we arrive into the causal relation for the  $T_n$  distributions:

$$T_n(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = T_m(x_1, \dots, x_m) T_{n-m}(x_{m+1}, \dots, x_n), \quad (3.5)$$

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<sup>3</sup>The test function  $g$  plays the part of switching on (off) the interaction, thus  $g(x) \in [0, 1]$ . Hence, when the limit  $g \rightarrow 1$  is taken adiabatically, we have a free system.

if the inequality is satisfied  $\{x_1, \dots, x_m\} > \{x_{m+1}, \dots, x_n\}$ ,<sup>4</sup> then we can say that  $T_n$  is a *causal ordered product* distribution. Since the sign  $>$  is understood in *stricto sensu*, the distribution  $T_n$  cannot be expressed in terms of the well known Feynman time-ordering product:  $T_n(x_1, \dots, x_n) \neq \mathcal{T}[T_1(x_1) \cdots T_1(x_n)]$ , which is endowed with UV divergences.<sup>5</sup>

- *Relativistic invariance*, in general,  $\mathcal{U}$  is a symmetry if for two observers  $\mathcal{O}$  and  $\mathcal{O}'$ , which look to the same system, the measured transition probabilities are equal. If we consider a unique asymptotically free particle space  $\mathcal{F}$ , the symmetry  $\mathcal{U}$  can be represented by a single operator  $U : \mathcal{F} \rightarrow \mathcal{F}'$ . Then, for the S-matrix:  $S$  and  $S'$  observed by  $\mathcal{O}$  and  $\mathcal{O}'$  are, respectively, related as follows

$$S' = U S U^{-1}. \quad (3.6)$$

The Epstein-Glaser method considers that, for the deduction of each element of the perturbative series (3.1), it is sufficient to take as (symmetry) axioms the translational and Lorentz invariance:

$$U(1, a) S[g] U^{-1}(1, a) = S[g_a], \quad g_a(x) = g(x - a), \quad (3.7)$$

$$U(\Lambda, 0) S[g] U^{-1}(\Lambda, 0) = S[g_\Lambda], \quad g_\Lambda(x) = g(\Lambda^{-1}x), \quad (3.8)$$

where  $U(\Lambda, a)$  is the continuous unitary representation of the orthochronous Poincaré group. Furthermore, when we replace these relations into the series (3.1), we find that the operator-valued distributions  $T_n$  can be written as

$$T_n(x_1, x_2, \dots, x_n) = T_n(x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n), \quad (3.9)$$

whereas the Lorentz invariance implies that

$$U(\Lambda, 0) T_n(x_1, x_2, \dots, x_n) U^{-1}(\Lambda, 0) = T_n(\Lambda x_1, \Lambda x_2, \dots, \Lambda x_n). \quad (3.10)$$

- *Asymptotic conditions and interaction*, in this formalism only the free asymptotic fields acting on the Fock space  $\mathcal{F}$  are used in order to construct  $S[g]$ . Thus, for the GQED<sub>4</sub>, we shall consider the set of electromagnetic and spinor free fields:  $(A_\mu, \psi, \bar{\psi})$ . This axiom also says that the one-point distribution  $T_1(x)$  is proportional to the interaction Lagrangian. Thus, from Eq.(2.11), we have that

$$T_1(x) = ie : \bar{\psi}(x) \gamma^\mu \psi(x) : A_\mu(x), \quad (3.11)$$

$e$  is the normalized constant coupling.

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<sup>4</sup>Which means that  $x_j^0 > x_i^0$ , for  $j = 1, \dots, m$  and  $i = m+1, \dots, n$ .

<sup>5</sup>In the time-ordered products a Heaviside  $\theta$ -function is present, one can show that the product of this function with singular distributions is in fact a divergent quantity [27].

The inductive method starts with the initial data  $T_1$  and also with  $\tilde{T}_1$  (due to Eq.(3.3)). Then, from these initial data, we can find the 2-point distribution  $T_2$ . In general, the inductive method proposes to find the  $n$ -order term  $T_n$  from the set  $\{T_1, \dots, T_{n-1}, \tilde{T}_1, \dots, \tilde{T}_{n-1}\}$ . For this purpose, the Epstein-Glaser approach introduce a well-defined distributional product, such as the intermediate  $n$ -point distributions

$$A'_n(x_1, \dots, x_n) \equiv \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n), \quad (3.12)$$

$$R'_n(x_1, \dots, x_n) \equiv \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X), \quad (3.13)$$

where  $P_2$  are all partitions of  $\{x_1, \dots, x_{n-1}\}$  into the disjoint sets  $X, Y$  such that  $|X| = n_1 \geq 1$  and  $|Y| \leq n-2$ . Moreover, other important distributions are obtained when the sums in Eqs.(3.12) and (3.13) are now extended over all partitions  $P_2^0$ , including the empty set. These are the advanced and retarded distributions

$$A_n(x_1, \dots, x_n) \equiv \sum_{P_2^0} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n) \quad (3.14)$$

$$\begin{aligned} &= A'_n(x_1, \dots, x_n) + T_n(x_1, \dots, x_n), \\ R_n(x_1, \dots, x_n) &\equiv \sum_{P_2^0} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) \quad (3.15) \\ &= R'_n(x_1, \dots, x_n) + T_n(x_1, \dots, x_n). \end{aligned}$$

By causal properties, one may easily conclude that  $R_n$  and  $A_n$  have retarded and advanced support, respectively,

$$\text{Supp } R_n(x_1, \dots, x_n) \subseteq \Gamma_{n-1}^+(x_n), \quad \text{Supp } A_n(x_1, \dots, x_n) \subseteq \Gamma_{n-1}^-(x_n), \quad (3.16)$$

where  $\Gamma_{n-1}^\pm(x_n) = \{(x_1, \dots, x_n) / x_j \in \bar{V}^\pm(x_n), \quad \forall j = 1, \dots, n-1\}$ , and  $\bar{V}^\pm(x_n)$  is the closed forward (backward) cone. These two distributions are not determined by the induction assumption, rather they are obtained by the *splitting* process of the so-called *causal distribution* defined as

$$D_n(x_1, \dots, x_n) \equiv R'_n(x_1, \dots, x_n) - A'_n(x_1, \dots, x_n) = R_n(x_1, \dots, x_n) - A_n(x_1, \dots, x_n). \quad (3.17)$$

For the case of GQED<sub>4</sub> we can write  $D_n$  as follows

$$D_n(x_1, \dots, x_n) = \sum_k d_n^k(x_1, \dots, x_n) : \prod_j \bar{\psi}(x_j) \prod_l \psi(x_l) \prod_m A(x_m) :, \quad (3.18)$$

where  $d_n^k(x_1, \dots, x_n)$  is the numerical part of the distribution. Besides, by translational invariance, we see that  $d_n^k$  depend only on relative coordinates:

$$d(x) \equiv d_n^k(x_1 - x_n, \dots, x_{n-1} - x_n) \in \mathcal{J}'(\mathbb{R}^m), \quad m = 4(n-1). \quad (3.19)$$



As aforementioned an important step in the analysis is the splitting of the numerical causal distribution  $d$  at the origin:  $\{x_n\} = \Gamma_{n-1}^+(x_n) \cap \Gamma_{n-1}^-(x_n)$ , into the advanced and retarded parts. These distributions are denoted as  $a$  and  $r$ , respectively. When we analyse the convergence of the sequence  $\{\langle d, \phi_\alpha \rangle\}$ , where  $\phi_\alpha$  has a decreasing support when  $\alpha \rightarrow 0^+$  and belongs to the Schwartz space  $\mathcal{S}$ , we find some natural distributional definitions. For instance, we name  $d$  as a distribution of singular order  $\omega$  if its Fourier transform  $\hat{d}(p)$  has a quasi-asymptotic  $\hat{d}_0(p) \neq 0$  at  $p = \infty$  with regard to a positive continuous function  $\rho(\alpha)$ ,  $\alpha > 0$ , i.e. if the limit

$$\lim_{\alpha \rightarrow 0^+} \rho(\alpha) \left\langle \hat{d}\left(\frac{p}{\alpha}\right), \phi(p) \right\rangle = \langle \hat{d}_0(p), \phi(p) \rangle \neq 0, \quad (3.20)$$

exists in  $\mathcal{S}'(\mathbb{R}^m)$ , with the *power-counting* function  $\rho(\alpha)$  satisfying

$$\lim_{\alpha \rightarrow 0} \frac{\rho(a\alpha)}{\rho(\alpha)} = a^\omega, \quad \forall a > 0, \quad (3.21)$$

or, equivalently,

$$\rho(\alpha) \rightarrow \alpha^\omega L(\alpha), \text{ when } \alpha \rightarrow 0^+, \quad (3.22)$$

where  $L(\alpha)$  is a quasi-constant function at  $\alpha = 0$ . Of course, there is an equivalent definition in the coordinate space, but, since the splitting process is more easily accomplished in the momentum space, this one suffices for our purposes. From this definition we have two distinct cases depending on the value of  $\omega$  [25], these are:

(i) *Regular* distributions - for  $\omega < 0$ , in this case the solution of the splitting problem is unique and the retarded distribution is defined by multiplying  $d$  by step functions, its form in the momentum space is given as follows

$$\hat{r}(p) = \frac{i}{2\pi} \text{sgn}(p^0) \int_{-\infty}^{+\infty} dt \frac{\hat{d}(tp)}{(1-t + \text{sgn}(p^0) i0^+)}, \quad (3.23)$$

identified as a dispersion relation without subtractions.

(ii) *Singular* distributions - for  $\omega \geq 0$ , then the solution cannot be obtained as in the *regular* case and, after a careful mathematical treatment, it may be shown that the retarded distribution is given by the central splitting solution

$$\hat{r}(p) = \frac{i}{2\pi} \text{sgn}(p^0) \int_{-\infty}^{+\infty} dt \frac{\hat{d}(tp)}{t^{\omega+1} (1-t + \text{sgn}(p^0) i0^+)}, \quad (3.24)$$

identified as a dispersion relation with  $\omega + 1$  subtractions. But in contrast with the regular case, this solution is not unique. If  $\hat{r}$  is a retarded part, then  $\tilde{r}$  defined as

$$\tilde{r}(p) = \hat{r}(p) + \sum_{a=0}^{\omega} C_a p^a, \quad (3.25)$$

is also a retarded part. The undetermined constants are not fixed by causality, and any other condition already introduced here, rather additional physical “normalization” conditions are necessary to fix them.

It is important to emphasize, however, that by the definition of the causal distribution (3.17), the singular order of the intermediate distributions, retarded and advanced parts, are in fact equal:

$$\omega(D) = \omega(R') = \omega(A') = \omega(R) = \omega(A). \quad (3.26)$$

Moreover, within some contexts, it shows to be useful to introduce the concept of graph in the causal approach. First, it should be clear that in this approach these are not Feynman integrals, the graphs are actually a purely schematic tool, so they do not play any part in the calculation itself. A class of graph must be understood here as the set of all elements of the S-matrix series (3.1) so that we have the same configuration. This means that, in our case, a particular class is such that the graphs have the same number of non-contracted spinor fields and/or the same number of non-contracted electromagnetic fields. Moreover, a graph is seen as an individual contribution. Then, the  $n$ -order distribution  $T_n$  can be written as the sum of  $n$ -order graphs of different classes

$$T_n = \sum_g T_n^g(x_1, x_2, \dots, x_n), \quad (3.27)$$

in this expression  $T_n^g$  contains graphs of the same class, and, for GQED<sub>4</sub>, each one of them can be written as follows

$$T_n^g(x) = \sum : \prod_{j=1}^{f_g} \bar{\psi}(x_{k_j}) t_g(x_1, x_2, \dots, x_n) \prod_{j=1}^{f_g} \psi(x_{n_j}) : : \prod_{j=1}^{l_g} A_{\mu_{m_j}}(x_{m_j}) :, \quad (3.28)$$

where  $l_g$  indicates the number of non-contracted electromagnetic fields,  $2f_g$  is the number of non-contracted spinors fields,  $t_g(x_1, x_2, \dots, x_n)$  is the contracted or numerical part and the sub-index  $g$  indicates a given fixed configuration. In particular, we define some important graphs:

1. The  $n$ -point lepton-lepton scattering graphs,

$$T_n^{LL}(x) = \sum : \prod_{j=1}^2 \bar{\psi}(x_{k_j}) t_g(x_1, x_2, \dots, x_n) \prod_{j=1}^2 \psi(x_{n_j}) : ; \quad (3.29)$$

2. The  $n$ -point lepton-photon scattering graphs,

$$T_n^{LP}(x) = \sum : \bar{\psi}(x_{k_1}) t_g(x_1, x_2, \dots, x_n) \psi(x_{n_1}) : : \prod_{j=1}^2 A_{\mu_{m_j}}(x_{m_j}) : ; \quad (3.30)$$

3. The  $n$ -point (fermionic) self-energy graphs,

$$T_n^{SE}(x) = \sum : \bar{\psi}(x_{k_1}) t_g(x_1, x_2, \dots, x_n) \psi(x_{n_1}) : ; \quad (3.31)$$

4. The  $n$ -point vacuum polarization graphs,

$$T_n^{VP}(x) = \sum : A_{\mu_{m_1}}(x_{m_1}) t_g(x_1, x_2, \dots, x_n) A_{\mu_{m_2}}(x_{m_2}) : . \quad (3.32)$$

## 4 Normalizability

The definition of normalizability in the causal perturbation theory is closely related to the singular order of each graph that contributes to the S-matrix series (3.1). In the previous section we have mentioned that if the retarded part is a singular distribution then it is not uniquely defined. This implies that for every graph of  $n$ -order  $T_n^g(x)$ , that has a singular order  $\omega \geq 0$ , a polynomial of degree  $\omega$  shall remain undetermined in momentum space. Hence, in each graph of  $n$ -order, with a finite number of free parameters, there are three possibilities when considering the inductive procedure:

(i) The number of free parameters increases with  $n$  without bound; then the model is called non-normalizable.

(ii) The total number of free parameters appearing in each order is finite; then the model is normalizable.

(iii) There is only a finite number of low-order graph with  $\omega \geq 0$ ; then the model is called super-normalizable.

In order to determine the normalizability of the GQED<sub>4</sub>, i.e. to determine the singular order of an arbitrary graph, we must first calculate the singular order of the contraction of two graphs via the inductive method.

### 4.1 Singular order of two contracted graphs

Following the inductive construction, we define the numerical part of the  $r$ -point graph  $G_1$  and the  $s$ -point graph  $G_2$  by the distributions

$$t_1(x_1 - x_r, \dots, x_{r-1} - x_r), \quad t_2(y_1 - y_s, \dots, y_{s-1} - y_s), \quad (4.1)$$

respectively. Besides, the graph that contributes to the S-matrix must be obtained after the splitting procedure, and then we can finally determine its singular order. But, as we have already mentioned in Sect. 3 (see Eq.(3.26)), a graph has the same singular order of a contracted graph (intermediate distribution formed by contractions). Therefore, for our purposes, it is enough to calculate the singular order of a contracted graph.

For instance, if we consider that the graphs  $G_1$  and  $G_2$  are contracted by  $\kappa$ -contractions of a same class of field; then, by taking translational invariance into account, the numerical part of the contracted graph takes the form:

$$t_1(x_1 - x_r, \dots, x_{r-1} - x_r) \prod_{j=1}^{\kappa} D_{a_{r_j} b_{s_j}}^{(+)}(x_{r_j} - y_{s_j}) t_2(y_1 - y_s, \dots, y_{s-1} - y_s), \quad (4.2)$$

where  $D_{a_{r_j} b_{s_j}}^{(+)}$  indicates the numerical part of the contraction between the points  $x_{r_j}$  and  $y_{s_j}$ . Also,  $a_{r_j}$  and  $b_{s_j}$  are the index of the associated fields into these points, respectively.

Moreover, we can write the numerical part of the contracted graph (4.2) using the relative variables

$$\xi_i = x_i - x_r \quad i = 1, \dots, r-1, \quad (4.3)$$

$$\eta_k = y_k - y_s \quad k = 1, \dots, s-1, \quad (4.4)$$

$$\eta = x_r - y_s, \quad (4.5)$$

so that we obtain

$$t(\xi_1, \dots, \xi_{r-1}, \eta_1, \dots, \eta_{s-1}, \eta) = t_1(\xi_1, \dots, \xi_{r-1}) \prod_{j=1}^{\kappa} D_{a_{r_j} b_{s_j}}^{(+)}(\xi_{r_j} - \eta_{s_j} + \eta) t_2(\eta_1, \dots, \eta_{s-1}). \quad (4.6)$$

In order to calculate the singular order of the contracted graph, we compute the Fourier transform of its numerical part,<sup>6</sup> which is given by the convolution

$$\begin{aligned} \hat{t}(p_1, \dots, p_{r-1}, q_1, \dots, q_{s-1}, q) &= \hat{t}_1(\dots, p_i, \dots, p_{r_j} - k_j, \dots) \hat{t}_2(\dots, q_k, \dots, q_{s_j} + k_j, \dots) \\ &\times (2\pi)^{-\frac{4\kappa}{2}} \int \prod_{j=1}^{\kappa} dk_j \delta\left(q - \sum_j k_j\right) \prod_{j=1}^{\kappa} \hat{D}_{a_{r_j} b_{s_j}}^{(+)}(k_j). \end{aligned} \quad (4.7)$$

Now we can calculate the power counting function  $\rho$  of the contracted graph by using the definition of the limit (3.20). Next, we need to compute the distribution  $\hat{t}(p)$ , which has the dimension  $m = 4(r+s-1)$ , evaluated in the test function  $\check{\phi}(\alpha p)$

$$\begin{aligned} \langle \hat{t}(p), \check{\phi}(\alpha p) \rangle &= \int d^{r-1} p d^{s-1} q d q \hat{t}_1(\dots, p_i, \dots, p_{r_j} - k_j, \dots) \hat{t}_2(\dots, q_k, \dots, q_{s_j} + k_j, \dots) \\ &\times (2\pi)^{-\frac{4\kappa}{2}} \int \prod_{j=1}^{\kappa} dk_j \delta\left(q - \sum_j k_j\right) \prod_{j=1}^{\kappa} \hat{D}_{a_{r_j} b_{s_j}}^{(+)}(k_j) \\ &\times \check{\phi}(\dots, \alpha p_i, \dots, \alpha p_{r_j}, \dots, \alpha q_k, \dots, \alpha q_{s_j}, \dots, \alpha q). \end{aligned} \quad (4.8)$$

It shows to be convenient for calculation purposes to consider the following change of variables

$$\alpha p_i \rightarrow p'_i, \quad \alpha p_{r_j} \rightarrow p'_{r_j} + k_j, \quad \alpha q_k \rightarrow q'_k, \quad \alpha q_{s_j} \rightarrow q'_{s_j} - k_j, \quad \alpha q \rightarrow q', \quad (4.9)$$

thus, after considering carefully this change into the integrating variables, we obtain that

$$\begin{aligned} \langle \hat{t}(p), \check{\phi}(\alpha p) \rangle &= \int \frac{d^{r-1} p'}{\alpha^{4(r-1)}} \frac{d^{s-1} q'}{\alpha^{4(s-1)}} \frac{d q'}{\alpha^4} \hat{t}_1\left(\dots, \frac{p'_i}{\alpha}, \dots, \frac{p'_{r_j}}{\alpha}, \dots\right) \hat{t}_2\left(\dots, \frac{q'_k}{\alpha}, \dots, \frac{q'_{s_j}}{\alpha}, \dots\right) \\ &\times (2\pi)^{-\frac{4\kappa}{2}} \int \frac{1}{\alpha^{4\kappa}} \prod_{j=1}^{\kappa} dk_j \delta\left(\frac{q' - \sum_j k_j}{\alpha}\right) \prod_{j=1}^{\kappa} \hat{D}_{a_{r_j} b_{s_j}}^{(+)}\left(\frac{k_j}{\alpha}\right) \\ &\times \check{\phi}\left(\dots, p'_i, \dots, p'_{r_j} + k_j, \dots, \dots, q'_k, \dots, q'_{s_j} - k_j, \dots, q'\right). \end{aligned} \quad (4.10)$$

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<sup>6</sup>A detailed calculation can be found in the Appendix B.

Taking into account that in 4-dimensions:  $\delta\left(\frac{k}{\alpha}\right) = \alpha^4 \delta(k)$ , and also the definition:

$$\lim_{\alpha \rightarrow 0^+} \hat{D}_{a_{r_j} b_{s_j}}^{(+)}\left(\frac{k}{\alpha}\right) = \alpha^{-\sigma} \hat{D}_{a_{r_j} b_{s_j}}^{0(+)}(k), \quad (4.11)$$

where  $\hat{D}_{a_{r_j} b_{s_j}}^{0(+)}$  is a nonvanishing distribution and  $\sigma$  is the singular order of  $\hat{D}_{a_{r_j} b_{s_j}}^{(+)}$ . Furthermore, introducing into Eq.(4.10) the power counting functions  $\rho_1(\alpha)$  and  $\rho_2(\alpha)$  associated to  $t_1$  and  $t_2$ , respectively, we can arrive into the expression:

$$\begin{aligned} & [\alpha^{-4} \rho_1(\alpha) \rho_2(\alpha)] \alpha^m \langle \hat{t}(p), \check{\phi}(\alpha p) \rangle = \\ & = \frac{1}{\alpha^{(4+\sigma)\kappa}} \int d^{r-1} p' d^{s-1} q' dq' \\ & \times \left[ \rho_1(\alpha) \hat{t}_1\left(\dots, \frac{p'_i}{\alpha}, \dots, \frac{p'_{r_j}}{\alpha}, \dots\right) \right] \left[ \rho_2(\alpha) \hat{t}_2\left(\dots, \frac{q'_k}{\alpha}, \dots, \frac{q'_{s_j}}{\alpha}, \dots\right) \right] \\ & \times (2\pi)^{-\frac{4\kappa}{2}} \int \prod_{j=1}^{\kappa} dk_j \delta\left(q' - \sum_j k_j\right) \prod_{j=1}^{\kappa} \hat{D}_{a_{r_j} b_{s_j}}^{(+)}(k) \\ & \times \check{\phi}\left(\dots, p'_i, \dots, p'_{r_j} + k_j, \dots, \dots, q'_k, \dots, q'_{s_j} - k_j, \dots, q'\right). \end{aligned} \quad (4.12)$$

Now we consider that the limits associated with the power counting functions of the graphs  $G_1$  and  $G_2$  exist and that they are given by:  $\lim_{\alpha \rightarrow 0^+} \rho(\alpha) \hat{t}_i\left(\frac{p}{\alpha}\right) = \hat{t}_{i,0}(p)$ . Then, we can take the limit:  $\alpha \rightarrow 0^+$  into the expression (4.12),

$$\begin{aligned} & \lim_{\alpha \rightarrow 0^+} [\alpha^{-4} \rho_1(\alpha) \rho_2(\alpha)] \alpha^m \langle \hat{t}(p), \check{\phi}(\alpha p) \rangle = \\ & = \alpha^{-(4+\sigma)\kappa} \int d^{r-1} p' d^{s-1} q' dq' \hat{t}_{1,0}\left(\dots, p'_i, \dots, p'_{r_j}, \dots\right) \hat{t}_{2,0}\left(\dots, q'_k, \dots, q'_{s_j}, \dots\right) \\ & \times (2\pi)^{-\frac{4\kappa}{2}} \int \prod_{j=1}^{\kappa} dk_j \delta\left(q' - \sum_j k_j\right) \prod_{j=1}^{\kappa} \hat{D}_{a_{r_j} b_{s_j}}^{0(+)}(k) \\ & \times \check{\phi}\left(\dots, p'_i, \dots, p'_{r_j} + k_j, \dots, \dots, q'_k, \dots, q'_{s_j} - k_j, \dots, q'\right). \end{aligned} \quad (4.13)$$

Besides, let us write the above expression in terms of the original variables (4.9), we have that the limit

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \left[ \alpha^{(4+\sigma)\kappa-4} \rho_1(\alpha) \rho_2(\alpha) \right] \hat{t}\left(\dots, \frac{p_i}{\alpha}, \dots, \frac{q_k}{\alpha}, \dots, \frac{q}{\alpha}\right) = \\ & = (2\pi)^{-\frac{4\kappa}{2}} \int \prod_{j=1}^{\kappa} dk_j \hat{t}_{1,0}\left(\dots, p_i, \dots, p_{r_j} - k_j, \dots\right) \hat{t}_{2,0}\left(\dots, q_k, \dots, q_{s_j} + k_j, \dots\right) \\ & \times \delta\left(q' - \sum_j k_j\right) \prod_{j=1}^{\kappa} \hat{D}_{a_{r_j} b_{s_j}}^{0(+)}(k), \end{aligned} \quad (4.14)$$

exist. Furthermore, by comparing (4.14) with the definition (3.20)

$$\lim_{\alpha \rightarrow 0^+} \rho(\alpha) \hat{t}\left(\frac{p}{\alpha}\right) = \hat{t}_0(p), \quad (4.15)$$

we can see that the power counting function  $\rho(\alpha)$  satisfies

$$\rho(\alpha) \sim \alpha^{(4+\sigma)\kappa-4} \rho_1(\alpha) \rho_2(\alpha) \sim \alpha^{(4+\sigma)\kappa-4+\omega(t_1)+\omega(t_2)}. \quad (4.16)$$

Finally, we can affirm that if the singular order of the graphs  $G_1$  and  $G_2$  are  $\omega(G_1)$  and  $\omega(G_2)$ , respectively, then the singular order of the contracted graph  $\omega(G)$  can be calculated by the formula

$$\omega(G) = \omega(G_1) + \omega(G_2) + (4 + \sigma) \kappa - 4. \quad (4.17)$$

recalling that  $\kappa$  is the number of contracted fields, and  $\sigma$  is the singular order of a single contraction. A simple generalization can be obtained when this process is applied to the contraction of different kind of quantum fields. Thus, in general, we find

$$\omega(G) = \omega(G_1) + \omega(G_2) + \sum_j (4 + \sigma_j) \kappa_j - 4, \quad (4.18)$$

where  $\kappa_j$  is the number of contractions of the type  $j$  with singular order  $\sigma_j$  associated to a particular class of quantum field.

For GQED<sub>4</sub> we have that there are two types of contractions:

- (i) The fermionic whose contractions  $\hat{D}_{a_r j b_{s_j}}^{1/2(+)} = S^{(\pm)}$  have singular order  $\omega(S^{(\pm)}) = -1$ ;
- (ii) The electromagnetic whose contraction  $\hat{D}_{a_r j b_{s_j}}^{(+)} = \hat{D}_{\mu_r j \nu_{s_j}}^{(+)}$  has singular order  $\omega(\hat{D}_{\mu_r j \nu_{s_j}}^{(+)}) = -4$ .

Thus, if the GQED<sub>4</sub> graphs  $G_1$  and  $G_2$  (with singular order  $\omega(G_1)$  and  $\omega(G_2)$ , respectively) are contracted by  $F$  spinorial and  $L$  electromagnetic contractions, we have that the contracted graph  $G$  has singular order:

$$\omega(G) = \omega(G_1) + \omega(G_2) + 3F - 4. \quad (4.19)$$

It should be remarked that the difference between the singular order of the electromagnetic propagator of QED<sub>4</sub> and GQED<sub>4</sub> is rather relevant, because in GQED<sub>4</sub> the singular order  $\omega(\hat{D}_{\mu\nu})$  cancels exactly the value of the dimension frame considered [22].<sup>7</sup> This is exactly the reason why the singular order of the contracting graph, in GQED<sub>4</sub>, does not depend on the number of electromagnetic contractions.

## 4.2 Super-normalizability proof

Now we want to calculate the singular order of any graph of the GQED<sub>4</sub>. The singular order of a graph is defined as the power of its associated power counting function and, in general, a graph can be obtained by the contraction of other graphs. Since the singular order of a contraction is given by a linear combination (see Eq.(4.19)), we can suppose that the singular order of a graph can be given by a linear combination of the number of non-contracted spinor fields  $f$  and the number of non-contracted electromagnetic fields  $l$ , and that it also depends on the perturbative order  $n$ , in the following form

$$\omega_n = an + bf + cl + d, \quad (4.20)$$

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<sup>7</sup>A similar analysis had been presented to show the renormalizability for higher-derivative quantum gravity [7].

where  $a$ ,  $b$ ,  $c$ , and  $d$  are constants that we must determine. If this argument is valid, then the graphs with  $n_{1,2}$ -point,  $f_{1,2}$  non-contracted fermionic and  $l_{1,2}$  non-contracted electromagnetic fields have the singular order:

$$\omega_{n_1} = an_1 + bf_1 + cl_1 + d, \quad (4.21)$$

$$\omega_{n_2} = an_2 + bf_2 + cl_2 + d, \quad (4.22)$$

respectively. From the result (4.19) we can affirm that the singular order of the contracted graph of  $n = n_1 + n_2$  order, with  $F$  fermionic contractions and  $L$  electromagnetic contractions, is given by

$$\omega_n = an + b(f_1 + f_2) + c(l_1 + l_2) + 2d + 3F - 4, \quad (4.23)$$

also, by a hypothesis present in (4.20), the singular order is equal to

$$\omega_n = an + b(f_1 + f_2 - 2F) + c(l_1 + l_2 - 2L) + d. \quad (4.24)$$

By comparing these relations we have that  $d = 4$ ,  $b = -\frac{3}{2}$  and  $c = 0$ . Then the singular order of the contracted graph is determined as

$$\omega_n = 4 + an - \frac{3}{2}f. \quad (4.25)$$

In order to fix the constant  $a$  we need to consider a known case. For this purpose, we may consider the results from Ref. [22], where we have found that the graph associated to the Bhabha scattering, i.e.  $f = 4$  and  $n = 2$ , has singular order  $-4$ . Hence, we can conclude that  $a = -1$ . Finally, we find that the singular order of a  $n$ -point graph with  $f$  and  $l$  non-contracted spinor and electromagnetic fields, respectively, has the singular order:<sup>8</sup>

$$\omega_n = 4 - n - \frac{3}{2}f. \quad (4.26)$$

A direct conclusion of this result is that for graphs with more than 5-point the singular order is negative. Therefore, from our initial definition, we can conclude that  $\text{GQED}_4$  is a *super-normalizable electrodynamics model*. Furthermore, we can determine the singular order of the lowest-order graphs:

1. The 4-point graphs have singular order:  $\omega_4 = -\frac{3}{2}f$ . Hence, the light-light scattering graph, the vacuum polarization graphs and the 4-point vacuum graph ( $f = 0$ ) all have singular order  $\omega = 0$ . Besides, the 4-point lepton-photon scattering graphs and the 4-point (fermionic) self-energy graphs ( $f = 2$ ) are regular distributions with singular order  $\omega = -3$ . Finally, the 4-point lepton-lepton scattering graphs ( $f = 4$ ) are also regular with singular order  $\omega = -6$ .
2. The 3-point graphs have singular order:  $\omega_3 = 1 - \frac{3}{2}f$ . Hence, the vertex graph with an odd number of non-contracted electromagnetic graphs ( $f = 0$ ) has singular order  $\omega = 1$ . But, from

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<sup>8</sup>There are interesting differences of the present result in comparison with the usual method [4], mainly because it does not depend on the external electromagnetic lines. The usual approach is reviewed in the appendix A.

Furry's theorem <sup>9</sup> these graphs do not contribute. Also, the usual vertex graph ( $f = 2$ ) is a regular distribution with singular order  $\omega = -2$ , then, in contrast with QED<sub>4</sub> results, this contribution is UV finite, as it was shown in Ref. [23].

3. The 2-point graphs have singular order:  $\omega_2 = 2 - \frac{3}{2}f$ . Hence, the one-loop vacuum polarization and the 2-point vacuum graph ( $f = 0$ ) have singular order  $\omega = 2$ . The 2-point lepton-photon scattering graph and the one-loop self-energy ( $f = 2$ ) are regular distributions with singular order  $\omega = -1$ . Again, in contrast with QED<sub>4</sub> results, the one-loop self-energy does not have UV divergences, as shown in Ref. [23]. The 2-point lepton-lepton scattering graph ( $f = 2$ ) is also regular, with singular order  $\omega = -4$ .
4. Finally we can assign to the fundamental vertex the singular order  $\omega_1 = 0$ . This is a practical assumption that allows us to deduce the singular order of graphs contracted with a fundamental vertex by Eq.(4.19).

## 5 Gauge invariance

To study the gauge structure of GQED<sub>4</sub> in the causal approach, we shall analyse how the S-matrix series

$$S[g] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 dx_2 \dots dx_n T_n(x_1, x_2, \dots, x_n) g(x_1) g(x_2) \dots g(x_n) \quad (5.1)$$

behave upon a gauge transformation. Since the causal approach only takes into account free fields, then we must formulate the gauge transformation on the electromagnetic free field ( $A_\mu$ ) and the free Dirac fields ( $\psi, \bar{\psi}$ ). However, the latter are not gauge invariant. Therefore, we only take into account the gauge transformation on the electromagnetic free field. Furthermore, we shall consider a gauge transformation by means of a *classical* (c-number) gauge function  $\Lambda(x)$

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \Lambda(x), \quad (5.2)$$

where  $\Lambda(x)$  vanishes at infinity and, since we consider the non-mixing gauge fixing condition Eq.(2.6), it satisfies the pseudo-differential equation:  $(1 + a^2 \square)^{1/2} \square \Lambda = 0$ . Our goal here is to show that the S-matrix is invariant by the gauge transformation (5.2). Hence, we need to calculate how each perturbative term is individually transformed.

### 5.1 Gauge transformation

In the same way as we did in our previous analysis, our discussion here will follow the inductive procedure. Thus, we start by analysing how the first perturbative term  $S_1[g] = \int dx T_1(x) g(x)$  changes

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<sup>9</sup>Following the same analysis developed for the causal approach of the QED<sub>4</sub> in Ref. [25], it is possible to show that the GQED<sub>4</sub> model does not break any discrete symmetries. Therefore, in particular, GQED<sub>4</sub> satisfies the Furry's theorem.



under the gauge transformation (5.2)

$$\begin{aligned} S'_1[g] = & S_1[g] + i \int dx \partial_\mu [ : \bar{\psi}(x) \gamma^\mu \psi(x) : ] \Lambda(x) g(x) \\ & + i \int dx : \bar{\psi}(x) \gamma^\mu \psi(x) : \Lambda(x) \partial_\mu g(x). \end{aligned} \quad (5.3)$$

Now, recalling the current conservation from the Dirac field equation

$$\partial_\mu [ : \bar{\psi}(x) \gamma^\mu \psi(x) : ] = 0, \quad (5.4)$$

and if we also consider that  $\partial_\mu g(x) = 0$ , we find that the term  $S_1[g]$  is gauge invariant. The last condition is naturally achieved when the adiabatic limit is considered,  $g \rightarrow 1$ , which represents the interacting physical system without external influences. Therefore, we shall study the gauge transformation of all perturbative contributions under the adiabatic limit,

$$S_n[1] = \frac{1}{n!} \int dx_1 dx_2 \dots dx_n T_n(x_1, x_2, \dots, x_n). \quad (5.5)$$

As aforementioned, we will only consider the transformation of the type (5.2) in this work. Thus, it is convenient to express the  $n$ -point distribution  $T_n$  in terms of the normally ordered product of photon operators

$$T_n(x_1, \dots, x_n) = \sum_{l=0}^n \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) : A_{\mu_1}(x_{k_1}) \dots A_{\mu_l}(x_{k_l}) : , \quad (5.6)$$

where  $\mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}$  contains the numerical part and the non-contracted spinor fields. Since  $\Lambda$  is a c-number, we can show that the electromagnetic contraction is gauge invariant

$$\left[ A_\mu'^{(-)}(x), A_\nu'^{(+) }(y) \right] = \left[ A_\mu^{(-)}(x), A_\nu^{(+)}(y) \right] = i D_{\mu\nu}^{(+)}(x-y). \quad (5.7)$$

Then we can immediately conclude that the quantity  $\mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}$  is also gauge invariant. Hence, the distribution  $T_n$  (5.6) has the following form under gauge transformation

$$T'_n(x_1, \dots, x_n) = \sum_{l=0}^n \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) : A'_{\mu_1}(x_{k_1}) \dots A'_{\mu_j}(x_{k_j}) \dots A'_{\mu_l}(x_{k_l}) : . \quad (5.8)$$

Now applying the gauge transformation (5.2) into the field  $A'_{\mu_j}$ , we can show that the transformed contribution  $S'_n \equiv S'_n[1]$  can be written as follows

$$\begin{aligned} S'_n = & \frac{1}{n!} \sum_{l=0}^n \int dx_1 \dots dx_n \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) : A'_{\mu_1}(x_{k_1}) \dots A_{\mu_j}(x_{k_j}) \dots A'_{\mu_l}(x_{k_l}) : \\ & + \frac{1}{n!} \sum_{l=0}^n \frac{1}{e} \int dx_1 \dots dx_n \left[ \partial_{\mu_j} \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) \right] : A'_{\mu_1}(x_{k_1}) \dots \Lambda(x_{k_j}) \dots A'_{\mu_l}(x_{k_l}) : . \end{aligned} \quad (5.9)$$

Hence, we see that in order to achieve a gauge invariant theory, we need to analyse whether the condition

$$\frac{\partial}{\partial x_{k_j}^{\mu_j}} \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) = 0, \quad (5.10)$$

is satisfied  $\forall j, \forall l$  and  $\forall n$ .

The above condition is now analyzed through the inductive method. Taking into account the first-order element,  $\mathcal{T}_1^\mu(x_1) = ie: \bar{\psi}(x) \gamma^\mu \psi(x) :$ , we easily see that

$$\partial_\mu \mathcal{T}_1^\mu(x_1) = 0, \quad (5.11)$$

where we have used the result (5.4). Besides, for higher-order terms we consider the inductive hypothesis, i.e. the condition (5.10) is valid  $\forall m \leq n-1$ , this means that

$$\frac{\partial}{\partial x_{k_j}^{\mu_j}} \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_m) = 0, \quad \forall j, \quad \forall l, \quad \text{and} \quad \forall m \leq n-1. \quad (5.12)$$

Thus, we find that the  $n$ -order intermediate distributions valued in the adiabatic limit

$$\int dx_1 \dots dx_n R'_n(x_1, \dots, x_n) = \int dx_1 \dots dx_n \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X), \quad (5.13)$$

$$\int dx_1 \dots dx_n A'_n(x_1, \dots, x_n) = \int dx_1 \dots dx_n \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n), \quad (5.14)$$

are gauge invariant. Hence, it follows from the definition (3.17) that the causal distribution  $D_n$  valued in the adiabatic limit is gauge invariant as well. Furthermore, if we write  $D_n$  as

$$D_n(x_1, \dots, x_n) = \sum_{l=0}^n \mathcal{D}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) : A_{\mu_1}(x_{k_1}) \dots A_{\mu_l}(x_{k_l}) : , \quad (5.15)$$

one can show that

$$\frac{\partial}{\partial x_{k_j}^{\mu_j}} \mathcal{D}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) = 0, \quad \forall j, \quad \forall l. \quad (5.16)$$

Thus, since  $T_n$  is defined such as:  $T_n = R_n - R'_n$ , we need to analyse now how the distribution  $R_n$  is transformed. Then, if the retarded distribution  $R_n$  is written as

$$R_n(x_1, \dots, x_n) = \sum_{l=0}^n \mathcal{R}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) : A_{\mu_1}(x_{k_1}) \dots A_{\mu_l}(x_{k_l}) : , \quad (5.17)$$

and the condition

$$\frac{\partial}{\partial x_{k_j}^{\mu_j}} \mathcal{R}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) = 0, \quad \forall j, \quad \forall l, \quad (5.18)$$

is satisfied, then the gauge invariance of  $R_n$  is automatically proved. Moreover, since  $R_n$  is the retarded part of  $D_n$  we have that

$$\mathcal{R}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l} = \begin{cases} \mathcal{D}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l} & \text{on } \Gamma_{n-1}^+(x_n) / \{(x_1, \dots, x_n)\} \\ 0 & \text{on } (\Gamma_{n-1}^+(x_n))^C, \end{cases} \quad (5.19)$$

where  $C$  denotes the complement. From this result we find that the condition (5.18) is almost satisfied

$$\frac{\partial}{\partial x_{k_j}^{\mu_j}} \mathcal{R}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) = 0, \quad \text{on } (\Gamma_{n-1}^+(x_n) / \{(x_1, \dots, x_n)\}) \cup (\Gamma_{n-1}^+(x_n))^C. \quad (5.20)$$

Thus, the distribution  $\frac{\partial}{\partial x_{k_j}^{\mu_j}} \mathcal{R}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}$  has causal support at the origin. Furthermore, since  $R'_n$  is gauge invariant, and by the property  $T_n = R_n - R'_n$ , we can conclude that the distribution  $\frac{\partial}{\partial x_{k_j}^{\mu_j}} \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}$  has also causal support at the origin.

It is interesting to remark that we can write  $\mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}$  as the contribution of all possible configuration of non-contracted spinor fields

$$\mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) = \sum_{f=0}^{2n} \left[ {}^f \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) \right], \quad (5.21)$$

where  ${}^f \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}$  contains the numerical part and the non-contracted spinor fields

$${}^f \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l}(x_1, \dots, x_n) \equiv \sum_{gl} : \prod_{j=1}^{f_g} \bar{\psi}(x_{i_j}) \left[ \mu_{m_1} \dots \mu_{m_l} t_{n_1 \dots n_{f_g}}^{i_1 \dots i_{f_g}}(x_1, x_2, \dots, x_n) \right] \prod_{j=1}^{f_g} \psi(x_{n_j}) : , \quad (5.22)$$

where the index  $\{k_1 \dots k_l\}$  are implicitly on the right-hand side of the above expression. Finally, since each class of graphs is independent, we have that each one of these contributions is also individually independent. Therefore, each contribution has causal support at the origin and by theorem <sup>10</sup> they have the form

$$\frac{\partial}{\partial x_{k_j}^{\mu_j}} \left[ {}^f \mathcal{T}_{k_1 \dots k_l}^{\mu_1 \dots \mu_l} \right] = \sum_{gf,l} : \prod_{j=1}^f \bar{\psi}(x_{i_j}) \left[ \sum_{|a|=0}^{\omega(g_{f,l})+1} C_a^{gf,l} D^a \delta(x_1 - x_n) \dots \delta(x_{n-1} - x_n) \right] \prod_{j=1}^f \psi(x_{n_j}) : , \quad (5.23)$$

where the summation is over all possible graphs with  $2f$  and  $l$  non-contracted spinor and electromagnetic fields, respectively. In particular, is considered that each graph  $g_{f,l}$  has singular order  $\omega(g_{f,l})$ .

Finally, the analysis of the gauge structure of GQED<sub>4</sub> was reduced to the study of the expression (5.23). Moreover, one can easily see that only graphs with singular order  $\omega(g_{f,l}) \geq -1$  contribute. We can therefore conclude that from the results of the previous section, only graphs with less than 6-point shall contribute for the analysis. Each nonzero case is usually called an anomaly, because it violates gauge invariance. Now, we analyse individually each possible case:

1. The 5-point graphs have singular order:  $\omega_5 = -1 - \frac{3}{2}f$ . Hence, graphs containing only non-contracted electromagnetic fields ( $f=0$ ) and with singular order  $\omega_5 = -1$  may be an anomaly. However, by Furry's theorem, these graphs do not contribute.
2. The 4-point graphs have singular order:  $\omega_4 = -\frac{3}{2}f$ . Hence, the light-light scattering and the vacuum polarization graphs,<sup>11</sup> each one with  $f=0$ , have singular order  $\omega=0$ . We analyse each case separately:

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<sup>10</sup>If the support of a distribution  $f \in \mathcal{D}'(\supset \mathcal{J}')$  contains a single point  $x=0$ , then it is unambiguously represented in the form:  $f(x) = \sum_{b=0}^{\omega} c_b D^b \delta(x)$ , where  $\omega$  is the singular order of  $f$ . This holds equivalently in the Fourier transformed space:  $\hat{f}(p) = \sum_{b=0}^{\omega} c_b p^b$ . For further detail see [37].

<sup>11</sup>The 4-point vacuum graphs do not have external electromagnetic fields, thus, it is not considered.

In the vacuum graph all electromagnetic field are contracted; besides, since this type of contraction is gauge invariant, this graph is too.

We can assume that the 4-point vacuum polarization graphs have the non-contracted electromagnetic fields in  $x_1$  and  $x_4$ , we have that

$${}^0\mathcal{T}_{VP}^{\mu_1\mu_4}(x_1, \dots, x_4) = \sum_{g_{0,2}} t^{\mu_1\mu_4}(x_1, x_2, x_3, x_4) \equiv \Pi^{\mu_1\mu_4}(x_1, x_2, x_3, x_4), \quad (5.24)$$

is a numerical distribution. Furthermore, from Eq.(5.23), we find that the derivative with respect to  $x_1$  is given by

$$\partial_{\mu_1} [\Pi^{\mu_1\mu_4}(x_1, \dots, x_4)] = \sum_{k=1}^3 C_{4k} \frac{\partial}{\partial (x_k)_{\mu_4}} [\delta(x_1 - x_4) \cdots \delta(x_3 - x_4)], \quad (5.25)$$

where  $\partial^{\mu_j} \equiv \frac{\partial}{\partial (x_j)_{\mu_j}}$ . We also omit the constant term because it is not Lorentz invariant. Moreover, since  $\partial_{\mu_1} \partial_{\mu_4} [\Pi^{\mu_1\mu_4}]$  is invariant with respect to the exchange  $x_1 \leftrightarrow x_4$ , we obtain

$$\partial_{\mu_1} [\Pi^{\mu_1\mu_4}(x_1, \dots, x_4)] = \partial_{\mu_1} [C g^{\mu_1\mu_4} \delta(x_1 - x_4) \cdots \delta(x_3 - x_4)], \quad (5.26)$$

a similar relation can be obtained for the derivative with respect to  $x_4$ . Thus, we conclude that the quantity between the square bracket is a distribution of singular order  $\omega(\Pi^{\mu_1\mu_4}) = 0$ . Hence, the initial impression of an anomaly can be removed by a normalization of  $\Pi^{\mu_1\mu_4}$ . Finally, we show that the 4-point vacuum polarization is a gauge invariant distribution and a transversal tensor

$$\partial_{\mu_j} \Pi^{\mu_1\mu_4}(x_1, x_2, x_3, x_4) = 0, \quad j = 1, 4. \quad (5.27)$$

For the 4-point light-light scattering graph, which has all the spinor field contracted, we have that the quantity

$${}^0\mathcal{T}_{LL}^{\mu_1 \dots \mu_4}(x_1, \dots, x_4) = t^{\mu_1 \dots \mu_4}(x_1, x_2, x_3, x_4), \quad (5.28)$$

is a numerical distribution. Hence, from Eq.(5.23), we have that its derivative with respect to  $x_1$  is given by

$$\begin{aligned} \partial_{\mu_1} [{}^0\mathcal{T}_{LL}^{\mu_1 \dots \mu_4}(x_1, \dots, x_4)] &= \left[ \sum_{k=1}^3 (C_{2k} \partial_k^{\mu_2} g^{\mu_3\mu_4} + C_{3k} \partial_k^{\mu_3} g^{\mu_2\mu_4} + C_{4k} \partial_k^{\mu_4} g^{\mu_2\mu_3}) \right] \\ &\times \delta(x_1 - x_4) \cdots \delta(x_3 - x_4). \end{aligned} \quad (5.29)$$

Since  $\partial_{\mu_1} \partial_{\mu_2} \partial_{\mu_3} \partial_{\mu_4} [{}^0\mathcal{T}_{LL}^{\mu_1 \dots \mu_4}]$  is symmetric with respect to the permutation of the derivative variables, we have that

$$\begin{aligned} \partial_{\mu_1} [{}^0\mathcal{T}_{LL}^{\mu_1 \dots \mu_4}(x_1, \dots, x_4)] &= \partial_{\mu_1} [C (g^{\mu_1\mu_2} g^{\mu_3\mu_4} + g^{\mu_1\mu_3} g^{\mu_2\mu_4} + g^{\mu_1\mu_4} g^{\mu_2\mu_3})] \\ &\times \delta(x_1 - x_4) \cdots \delta(x_3 - x_4). \end{aligned} \quad (5.30)$$

A similar relation can be obtained for the other derivative variables. Like in the previous case, the anomaly can be removed by a normalization of  ${}^0\mathcal{T}_{LL}^{\mu_1\cdots\mu_4}$ . Therefore, the 4-point light-light scattering graph is a gauge invariant distribution and transversal tensor

$$\partial_{\mu_j} t^{\mu_1\cdots\mu_4}(x_1, x_2, x_3, x_4) = 0, \quad j = 1, 2, 3, 4. \quad (5.31)$$

3. The 3-point graphs have singular order:  $\omega_3 = 1 - \frac{3}{2}f$ . Hence, the only possible anomaly are those vertex graphs with three and one non-contracted electromagnetic graphs ( $f = 0$ ), both graphs have singular order  $\omega = 1$ ; but, according to the Furry's theorem these graphs do not contribute.
4. The 2-point graphs have singular order:  $\omega_2 = 2 - \frac{3}{2}f$ . However, the 2-point vacuum graph ( $\omega = 2$ ), the one-loop self-energy ( $\omega = -1$ ) and the lepton-lepton scattering graph ( $\omega = -1$ ) do not have external electromagnetic fields. Besides, since this type of contraction is gauge invariant, these graphs are gauge invariant as well.

The one-loop vacuum polarization has singular order  $\omega = 2$ , and following the procedure as in Ref. [25], it can be transformed by a normalization into a gauge invariant distribution.

The 2-point lepton-photon scattering graph is a regular distribution with singular order  $\omega = -1$ . Since this graph has two contributions, with help of (5.23), we have that

$$\begin{aligned} \partial_{\mu_1} [{}^2\mathcal{T}_{Com}^{\mu_1\mu_2}(x_1, x_2)] &= C_{12} : \bar{\psi}(x_1) \delta(x_1 - x_2) \gamma^{\mu_2} \psi(x_2) : \\ &+ C_{21} : \bar{\psi}(x_2) \delta(x_2 - x_1) \gamma^{\mu_2} \psi(x_1) : . \end{aligned} \quad (5.32)$$

Further, using the Dirac equation for the fermionic propagator  $S^F$ , the above formula can be transformed as follows

$$\begin{aligned} \partial_{\mu_1} [{}^2\mathcal{T}_{Com}^{\mu_1\mu_2}(x_1, x_2)] &= i\partial_{\mu_1} \left[ -C_{12} : \bar{\psi}(x_1) \gamma^{\mu_1} S^F(x_1 - x_2) \gamma^{\mu_2} \psi(x_2) : \right. \\ &\left. + C_{21} : \bar{\psi}(x_2) \gamma^{\mu_2} S^F(x_2 - x_1) \gamma^{\mu_1} \psi(x_1) : \right], \end{aligned} \quad (5.33)$$

the quantity on the square bracket is a distribution of singular order  $\omega({}^2\mathcal{T}_{Com}^{\mu_1\mu_2}) = -1$ . If we recall to the invariance under charge conjugation, we have that  $C_{12} = -C_{21}$ . Hence, since the quantity  $\partial_{\mu_1} [{}^2\mathcal{T}_{Com}^{\mu_1\mu_2}(x_1, x_2)]$  has support on  $x_1 = x_2$ , we can conclude that this anomaly can be removed. One may obtain the same result when considering the derivative with respect to  $x_2$ . Finally, we find that

$$\partial_{\mu_j} [{}^2\mathcal{T}_{Com}^{\mu_1\mu_2}(x_1, x_2)] = 0, \quad j = 1, 2, \quad (5.34)$$

which means that the lepton-photon scattering graphs are gauge invariant.

## 5.2 Ward-Takahashi-Fradkin identities

The Ward-Takahashi-Fradkin (WTF) identities are usually derived in QED<sub>4</sub> scattering processes when free photons are involved [38]. In the framework of functional method, these (coupled) identities are satisfied by the complete Green's functions. In contrast with this non-perturbative method, the causal approach determines these relations perturbatively order-by-order. Using the results obtained previously, we will see that the WTF identities for GQED<sub>4</sub> are easily derived.

*Vacuum polarization transversality.-* The contribution to the vacuum polarization is a class of graphs with two non-contracted electromagnetic fields. The transversality condition must be proved for each graph of this class. Hence, for the  $n$ -order vacuum polarization graph we shall assume that the non-contracted electromagnetic fields are in  $(x_1, x_2)$ , then we have that

$${}^0\mathcal{T}_{VP}^{\mu_1\mu_2}(x_1, x_2, \dots, x_n) = \Pi^{\mu_1\mu_2}(x_1, x_2, \dots, x_n), \quad (5.35)$$

is a numerical distribution. Moreover, by taking into account Eq.(5.23), we have that

$$\partial_{\mu_j} \Pi^{\mu_1\mu_2}(x_1, x_2, \dots, x_n) = 0, \quad j = 1, 2, \quad \forall n > 4. \quad (5.36)$$

Besides, by a normalization, we have proved this relation for  $n = 2$  and  $n = 4$ ; therefore, the transversality of the vacuum polarization tensor is guaranteed

$$\partial_{\mu_j} \Pi^{\mu_1\mu_2}(x_1, x_2, \dots, x_n) = 0, \quad j = 1, 2, \quad \forall n \geq 2, \quad (5.37)$$

and, by relabelling the derivative variable we can show the validity of this relation for the points  $(x_3, x_4, \dots)$ .

*Vertex and self-energy.-* For a  $n$ -point vertex graph, we can choose the non-contracted electromagnetic field in  $x_n$  and the two non-contracted spinor fields in  $(x_i, x_j)$ . In general, we can recognize three different possible graphs: (i) when all non-contracted fields are in different points; (ii) when one of the non-contracted spinor is in the same point of the non-contracted electromagnetic field, so this vertex is connected to the graph by a spinor line; (iii) when the two non-contracted spinor fields are in the same point, then this vertex is connected to the graph by an electromagnetic line. Taking all these cases into account we have that

$$\begin{aligned} {}^2\mathcal{T}_{Ver}^{\mu_n}(x_1, x_2, \dots, x_n) = & \sum_{i \neq j}^{n-1} : \bar{\psi}(x_i) \Lambda^{\mu_n}(x_n, x_i, x_j, \dots) \psi(x_j) : \\ & + \sum_{i \neq j}^{n-1} : \bar{\psi}(x_n) \gamma^{\mu_n} S^F(x_n - x_i) \Sigma(x_i, x_j, \dots) \psi(x_j) : \\ & + \sum_{i \neq j}^{n-1} : \bar{\psi}(x_i) \Sigma(x_i, x_j, \dots) S^F(x_j - x_n) \gamma^{\mu_n} \psi(x_n) : \\ & + \sum_{i \neq j}^{n-1} : \bar{\psi}(x_i) \Pi^{\mu_n \mu_j}(x_n, x_j, \dots) D_{\mu_j \mu_i}^F(x_j - x_i) \gamma^{\mu_i} \psi(x_i) : . \end{aligned} \quad (5.38)$$

where  $\Lambda^{\mu_n}$  and  $\Sigma$  are the vertex function and the self-energy, respectively. Since the vertex function has singular order  $-2$ , it follows from Eq.(5.23) that

$$\partial_{\mu_n} \left[ {}^2\mathcal{T}_{Ver}^{\mu_n}(x_1, x_2, \dots, x_n) \right] = 0, \quad (5.39)$$

without requiring any assumption. Moreover, from the transversality condition for the vacuum polarization (5.37) and from the following identities

$$\frac{\partial}{\partial x_n^{\mu_n}} \left[ \bar{\psi}(x_n) \gamma^{\mu_n} S^F(x_n - x_i) \right] = i \bar{\psi}(x_n) \delta(x_n - x_i), \quad (5.40)$$

$$\frac{\partial}{\partial x_n^{\mu_n}} \left[ S^F(x_j - x_n) \gamma^{\mu_n} \psi(x_n) \right] = -i \delta(x_j - x_n) \psi(x_n), \quad (5.41)$$

we obtain the relation

$$\begin{aligned} \partial_{\mu_n} \left[ {}^2\mathcal{T}_{Ver}^{\mu_n}(x_1, x_2, \dots, x_n) \right] &= \sum_{i \neq j}^{n-1} : \bar{\psi}(x_i) \partial_{\mu_n} \Lambda^{\mu_n}(x_n, x_i, x_j, \dots) \psi(x_j) : \\ &\quad + i \sum_{i \neq j}^{n-1} : \bar{\psi}(x_n) \delta(x_n - x_i) \Sigma(x_i, x_j, \dots) \psi(x_j) : \\ &\quad - i \sum_{i \neq j}^{n-1} : \bar{\psi}(x_i) \Sigma(x_i, x_j, \dots) \delta(x_j - x_n) \psi(x_n) : . \end{aligned} \quad (5.42)$$

Finally, using the condition (5.39) into the above formula we obtain the closed relation

$$\partial_{\mu_n} \Lambda^{\mu_n}(x_n, x_i, x_j, \dots) = i \left[ \delta(x_j - x_n) \Sigma(x_i, x_j, \dots) - \delta(x_n - x_i) \Sigma(x_i, x_j, \dots) \right]. \quad (5.43)$$

Nonetheless, since  $\Sigma$  and  $\Lambda^{\mu_n}$  are translational invariant, we can rewrite  $\Lambda^{\mu_n}$  equivalently now with respect to  $x_n$  and  $\Sigma$  with respect to  $x_j$ . Hence, we arrive into the following Ward-Takahashi-Fradkin identity written in the configuration space

$$\partial_{\mu_n} \Lambda^{\mu_n}(x_i - x_n, x_j - x_n, \dots) = i \left[ \delta(x_j - x_n) \Sigma(x_i - x_j, \dots) - \delta(x_i - x_n) \Sigma(x_i - x_j, \dots) \right]. \quad (5.44)$$

Now, considering the change of variable  $y_k = x_k - x_n$

$$- \sum_k \partial_{y_k}^{\nu} \Lambda_{\nu}(y_i, y_j, \dots) = i \left[ \delta(y_j) \Sigma(y_i - y_j, \dots) - \delta(y_i) \Sigma(y_i - y_j, \dots) \right], \quad (5.45)$$

and taking the Fourier transform of this last relation, we obtain the desired form for the Ward-Takahashi-Fradkin identity written in the momentum space

$$\begin{aligned} (2\pi)^2 \left( \sum_{k=1}^{n-1} p_k^{\nu} \right) \hat{\Lambda}_{\nu}(p_i, p_j, \dots) &= \hat{\Sigma}(p_i, \dots, p_{j-1}, p_{j+1}, \dots) \\ &\quad - \hat{\Sigma} \left( - \left( \sum_{k \neq i} p_k \right), \dots, p_{j-1}, p_{j+1}, \dots \right). \end{aligned} \quad (5.46)$$

In particular, let us consider the case  $n = 3$ ,  $i = 1$  and  $j = 2$ ,

$$(p_1 + p_2)_\nu \hat{\Lambda}^\nu(p_1, p_2) = (2\pi)^{-2} [\hat{\Sigma}(p_1) - \hat{\Sigma}(-p_2)]. \quad (5.47)$$

Besides, relabelling the momenta such as:  $p_1 \rightarrow p$  and  $p_2 \rightarrow -q$ , then  $\hat{\Lambda}^\nu(p, -q) \rightarrow \hat{\Lambda}^\nu(p, q)$ , we arrive into the well-known form of the one-loop Ward-Takahashi-Fradkin identity

$$(p - q)_\nu \hat{\Lambda}^\nu(p, q) = (2\pi)^{-2} [\hat{\Sigma}(p) - \hat{\Sigma}(q)]. \quad (5.48)$$

Finally, taking the limit  $p \rightarrow q$ , we find the following relation

$$\hat{\Lambda}^\nu(p, p) = (2\pi)^{-2} \frac{\partial}{\partial p_\nu} \hat{\Sigma}(p). \quad (5.49)$$

This relation may be used to find the one-loop vertex (at zero transferred momentum) from the one-loop self-energy expression.

## 6 Conclusion

In this paper we have discussed the normalizability and gauge invariance of  $\text{GQED}_4$  theory in the context of causal perturbation theory, as proposed by Epstein and Glaser. These general physical properties are not considered as axioms in this formalism, but their validity has been showed via an inductive procedure provided by the Epstein-Glaser causal method.

First, we remark that if the normalizability of  $\text{GQED}_4$  were analysed following the usual approach, since the ('bare') fundamental vertex of  $\text{QED}_4$  and  $\text{GQED}_4$  are the same then we would naively conclude that these models must have the same (perturbative) normalizability nature. However, when the singular order of a (complete) graph is properly analysed by the causal approach, the singular order of the internal lines (propagators) is taken into account and plays a crucial part in the analysis. Thus, an analysis of graphs with internal photon lines showed that they must have lower singular order than their  $\text{QED}_4$  counterparts. The singular order formula (4.26), obtained for any graph, indicates that only graphs with less than 5-point may be UV divergent. Therefore, we can conclude that  $\text{GQED}_4$  is a super-normalizable theory.

Furthermore, the general expression for the singular order allowed us to prove that the  $\text{GQED}_4$  is almost an explicitly gauge invariant theory. This is because only graphs with no more than 5-point may violate gauge invariance. We have proved in detail that all these possible anomalies can suitably be removed by a normalization. In particular, the transversality property is satisfied for the cases: the two- and four-point vacuum polarization graphs, the 4-point light-light scattering graph. This strong result led us to prove straightforwardly the Ward-Takahashi-Fradkin identities. Both the vacuum polarization transversality and the relation between the vertex and the self-energy are trivially proven within this framework.

In conclusion, we have proved that the generalized quantum electrodynamics is almost an ultraviolet finite field theory, and divergences appear only at lower perturbative order. Thus, for futures works



we intend to compute, via the causal approach, the one-loop radiative correction functions: vacuum polarization, self-energy and vertex function, and find explicitly their singular order which must be in accordance with the result obtained in section 4.

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## A Degree of divergence of GQED<sub>4</sub> graphs

Let us now discuss further methods to determine the degree of divergence of a graph; with particular interest in each method prediction that can be used as an approach for comparison of accuracy. For this purpose, we consider first the following interaction term for GQED<sub>4</sub>

$$\mathcal{L}_{int} = e : \bar{\psi}(x) \gamma^\mu \psi(x) : A_\mu(x), \quad (\text{A.1})$$

in this vertex we have that the number of fermionic lines is  $n_F = 2$ , and the number of bosonic lines is  $n_L = 1$ . The general definition of the superficial degree of divergence  $w$ , of an arbitrary connected one-particle irreducible Feynman diagram  $\Gamma$ , is the actual degree of divergence of the integration over the region of momentum space in which the momenta of all internal lines go to infinity together. This is the number of factors of momentum in the numerator minus the number in the denominator of the integrand, plus four for every independent four-momentum over which we integrate.

To calculate  $w$ , we will need to know the following detail about the diagram:

$$\begin{aligned} I_F &\equiv \text{number of internal fermionic lines,} \\ I_L &\equiv \text{number of internal electromagnetic lines,} \\ f &\equiv \text{number of external fermionic lines,} \\ l &\equiv \text{number of external electromagnetic lines,} \\ n &\equiv \text{number of vertices of interaction.} \end{aligned}$$

The asymptotic behaviour of the fermionic free Feynman propagator is given by

$$\hat{S}^F(p) = (2\pi)^{-2} \frac{(\gamma \cdot p + m)}{p^2 - m^2 + i0^+} \sim (\gamma \cdot p)^{-1}, \quad (\text{A.2})$$

thus, it has the power momentum

$$w_F = -1, \quad (\text{A.3})$$

as the usual. However, for the Bopp-Podolsky electromagnetic free Feynman propagator ( $\xi = 1$ ):

$$\hat{D}_{\mu\nu}^F(k) = -(2\pi)^{-2} g_{\mu\nu} \left( \frac{1}{k^2 + i0^+} - \frac{1}{k^2 - m_a^2 + i0^+} \right) \sim m_a^2 k^{-4}, \quad (\text{A.4})$$

we have the power momentum

$$w_L = -4. \quad (\text{A.5})$$

Adding the contributions from the propagators and total number of independent momentum variables of integration, we have that

$$w(\Gamma_n) = I_F w_F + I_L w_L + 4[I_F + I_L - (n-1)], \quad (\text{A.6})$$

$$= 4 + I_F(w_F + 4) + I_L(w_L + 4) - 4n, \quad (\text{A.7})$$

using the topological identities  $2I_L + l = n$  and  $2I_F + f = 2n$ , we obtain that

$$w(\Gamma_n) = 4 - \left( f \frac{(w_F + 4)}{2} + l \frac{(w_L + 4)}{2} \right) + n \left( w_F + \frac{(w_L + 4)}{2} \right). \quad (\text{A.8})$$

Finally, replacing (A.3) and (A.5) into (A.8), we find that the degree of divergence of a  $n$ -point GQED<sub>4</sub> connected graph is given by

$$w(\Gamma_n) = 4 - \frac{3}{2}f - n, \quad (\text{A.9})$$

this result is identical to the one obtained by the causal approach (4.26).

Nonetheless, the degree of divergence of a connected graph can also be obtained by a simple dimensional analysis.<sup>12</sup> Considering that the action must be dimensionless in natural units ( $\hbar = c = 1$ ), so each term in the Lagrangian density must have length dimensionality +4,

$$\mathcal{L}_{GQED} = \mathcal{L}_D + \mathcal{L}_P + \mathcal{L}_{int}. \quad (\text{A.10})$$

Then, for the fermionic part,  $\mathcal{L}_D = \bar{\psi}(i\gamma \cdot \partial - m)\psi$ , it follows that

$$[\psi] = [\bar{\psi}] = 3/2. \quad (\text{A.11})$$

Besides, from the Bopp-Podolsky electromagnetic theory

$$\mathcal{L}_P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\mu F^{\mu\sigma}\partial^\nu F_{\nu\sigma} - \frac{1}{2\xi}(\partial \cdot A)(1 + a^2\Box)(\partial \cdot A), \quad (\text{A.12})$$

we have that

$$[A_\mu] = 1, \quad (\text{A.13})$$

and also  $[\xi] = 0$  and  $[a] = -1$ . Moreover, from the interaction part we obtain that the dimensionality for the coupling constant reads

$$[e] = 4 - 2[\psi] - [A_\mu]. \quad (\text{A.14})$$

In general, the free propagator of a field is a four-dimensional Fourier transform of the vacuum expectation value of a time-ordered product of a pair of those free fields. For instance, in GQED<sub>4</sub> we have the two propagators

$$(2\pi)^{-2} \int d^4x \langle 0 | T \{ \psi_a(x), \bar{\psi}_b(0) \} | 0 \rangle e^{ipx} = \hat{S}_{ab}^F(p), \quad (\text{A.15})$$

$$(2\pi)^{-2} \int d^4x \langle 0 | T \{ A_\mu(x), A_\nu(0) \} | 0 \rangle e^{ikx} = \hat{D}_{\mu\nu}^F(k). \quad (\text{A.16})$$

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<sup>12</sup>This method has the advantage that it does not consider the structure of Feynman diagrams.

From these very definitions we obtain the following relations

$$[\psi] = \frac{[\hat{S}^F] + 4}{2}, \quad [A_\mu] = \frac{[\hat{D}_{\mu\nu}^F] + 4}{2}. \quad (\text{A.17})$$

Replacing these results back into (A.14) we have that

$$[e] = 4 - ([\hat{S}^F] + 4) - \frac{([\hat{D}_{\mu\nu}^F] + 4)}{2}. \quad (\text{A.18})$$

Applying the same dimensional analysis for the time-ordered part of the  $n + f + l$  interaction Lagrangian formed by a  $n$ -point connected Feynman graph  $\Gamma_n$  with  $f$  external fermionic lines and  $l$  external electromagnetic lines, contracted with  $f + l$  vertices, we find that

$$[\Gamma_n] = 4 - f([\hat{S}^F] + 4 - [\psi]) - l([\hat{D}_{\mu\nu}^F] + 4 - [A_\mu]) - n[e]. \quad (\text{A.19})$$

Furthermore, using the results (A.17) it follows

$$[\Gamma_n] = 4 - f[\psi] - l[A_\mu] - n[e]. \quad (\text{A.20})$$

Finally, making use of previous results  $[\psi] = 3/2$ ,  $[A_\mu] = 1$  and  $[e] = 0$ , we find that the degree of divergence obtained from a dimensional analysis is

$$[\Gamma_n] = 4 - \frac{3}{2}f - l. \quad (\text{A.21})$$

In the usual approach it is assumed that  $[\Gamma_n] = w(\Gamma_n)$ , but we see that this clearly contradicts the previous result (A.9). However, if we take an alternative route on the analysis, by replacing instead (A.17) and (A.18) into (A.20) we arrive into

$$[\Gamma_n] = 4 - \left( f \frac{([\hat{S}^F] + 4)}{2} - l \frac{([\hat{D}_{\mu\nu}^F] + 4)}{2} \right) - n \left( 4 - ([\hat{S}^F] + 4) - \frac{([\hat{D}_{\mu\nu}^F] + 4)}{2} \right). \quad (\text{A.22})$$

We see that this formula is almost similar to (A.8). Actually, for QED<sub>4</sub> they are equivalent, but for GQED<sub>4</sub> the central problem is given by the fact that

$$[\hat{D}_{\mu\nu}^F] = 2 \neq w_L = 4. \quad (\text{A.23})$$

## B Distributional Fourier transform

A distribution  $T$  is a linear continuous functional defined in a space of rapidly decreasing test functions  $\{\varphi\}$ , the so-called Schwartz space ( $\mathcal{S}$ ),

$$T : \varphi \rightarrow \langle T, \varphi \rangle \in \mathbb{C}. \quad (\text{B.1})$$

In the space  $\mathcal{S}$  is possible to define the Fourier transform of a distribution. Then, the Fourier transformed distribution  $\hat{T}$  is formally defined

$$\hat{T} : \varphi \rightarrow \langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle, \quad (\text{B.2})$$

where  $\hat{\varphi}$  is the Fourier transformed test function. For instance, we can find that the Fourier transform of the one dimensional  $\delta$ -Dirac distribution is given by:  $\hat{\delta}(k) = (2\pi)^{-1/2}$ . Moreover, since the distributional Fourier transform satisfies the same properties of the usual function, in practice, we can write  $\hat{T}$ , defined in  $4n$ -dimension, as follows

$$\mathcal{F}[T(x)](p) = \hat{T}(p) = (2\pi)^{-2n} \int \prod_{j=1}^n dx_j T(x) \exp \left[ i \sum_{l=1}^n p_l \cdot x_l \right]. \quad (\text{B.3})$$

In order to illustrate this definition, let us consider the Fourier transform of the following distributional product

$$t(\xi_1, \dots, \xi_{r-1}, \eta_1, \dots, \eta_{s-1}, \eta) = t_1(\xi_1, \dots, \xi_{r-1}) \prod_{j=1}^{\kappa} D_{a_{r_j} b_{s_j}}^{(+)}(\xi_{r_j} - \eta_{s_j} + \eta) t_2(\eta_1, \dots, \eta_{s-1}). \quad (\text{B.4})$$

First, we define the conjugated variables so that  $\xi_i \rightarrow p_i$ ,  $\eta_k \rightarrow q_k$ , and  $\eta \rightarrow q$ , then the Fourier transform is given by

$$\begin{aligned} \hat{t}(p_1, \dots, p_{r-1}, q_1, \dots, q_{s-1}, q) &= (2\pi)^{-\frac{4(r+s-1)}{2}} \int d^{r-1} \xi d^{s-1} \eta d\eta \exp \left( i \sum_{k=1}^{r-1} p_k \xi_k + i \sum_{k=1}^{s-1} q_k \eta_k + q\eta \right) \\ &\times t_1(\xi_1, \dots, \xi_{r-1}) \prod_{j=1}^{\kappa} D_{a_{r_j} b_{s_j}}^{(+)}(\xi_{r_j} - \eta_{s_j} + \eta) t_2(\eta_1, \dots, \eta_{s-1}). \end{aligned} \quad (\text{B.5})$$

Besides, we have that the Fourier transform of the distribution  $D_{a_{r_j} b_{s_j}}^{(+)}$  is defined as

$$D_{a_{r_j} b_{s_j}}^{(+)}(\xi_{r_j} - \eta_{s_j} + \eta) = (2\pi)^{-2} \int dk_j \hat{D}_{a_{r_j} b_{s_j}}^{(+)}(k_j) \exp[-ik_j(\xi_{r_j} - \eta_{s_j} + \eta)], \quad (\text{B.6})$$

also by considering the identity

$$\begin{aligned} \sum_{i=1}^{r-1} p_i \xi_i + \sum_{k=1}^{s-1} q_k \eta_k + q\eta &= \sum_{i \neq r_j} p_i \xi_i + \sum_{k \neq s_j} q_k \eta_k + \sum_j (p_{r_j} - k_j) \xi_{r_j} + \sum_j (q_{s_j} + k_j) \eta_{s_j} \\ &+ \sum_j k_j (\xi_{r_j} - \eta_{s_j} + \eta) + \left( q - \sum_j k_j \right) \eta, \end{aligned} \quad (\text{B.7})$$

we can rewrite  $\hat{t}$ , Eq.(B.5), in the following form

$$\begin{aligned}
& \hat{t}(p_1, \dots, p_{r-1}, q_1, \dots, q_{s-1}, q) = \\
& = (2\pi)^{-\frac{4(r-1)}{2}} \int d^{r-1} \xi \exp i \left( \sum_{i \neq r_j} p_i \xi_i + \sum_j (p_{r_j} - k_j) \xi_{r_j} \right) t_1(\xi_1, \dots, \xi_{r-1}) \\
& \times (2\pi)^{-\frac{4(s-1)}{2}} \int d^{s-1} \eta \exp i \left( \sum_{k \neq s_j} q_k \eta_k + \sum_j (q_{s_j} + k_j) \eta_{s_j} \right) t_2(\eta_1, \dots, \eta_{s-1}) \\
& \times (2\pi)^{-\frac{4l}{2}} \int \prod_{j=1}^{\kappa} dk_j (2\pi)^{-\frac{4}{2}} \int d\eta \exp i \left[ \left( q - \sum_j k_j \right) \eta \right] \prod_{j=1}^{\kappa} \hat{D}_{a_{r_j} b_{s_j}}^{(+)}(k_j). \tag{B.8}
\end{aligned}$$

Finally, identifying the respective Fourier transform of  $t_1$  and  $t_2$ , this expression can be reduced to

$$\begin{aligned}
\hat{t}(p_1, \dots, p_{r-1}, q_1, \dots, q_{s-1}, q) &= \hat{t}_1(\dots, p_i, \dots, p_{r_j} - k_j, \dots) \hat{t}_2(\dots, q_k, \dots, q_{s_j} + k_j, \dots) \\
&\times (2\pi)^{-\frac{4\kappa}{2}} \int \prod_{j=1}^{\kappa} dk_j \delta \left( q - \sum_j k_j \right) \prod_{j=1}^{\kappa} \hat{D}_{a_{r_j} b_{s_j}}^{(+)}(k_j). \tag{B.9}
\end{aligned}$$

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